Inequality Analysis and Applications of Differential Equations

Differential equations have very broad applications in Biology, Chemistry, Economics, Engineering, Physics and Mathematics. Most differential equations arising from applications cannot be solved in terms of elementary functions and integral forms. Therefore, instead of solving these types equations, researchers investigate existence and properties of solutions and numerical solutions. This research will be in the following three directions.

0.0.1 Wazewski-Type Upper and Lower Bounds of Solutions of Differential Equations

Although students learn how to solve a system of linear differential equations with real constant coefficients,$X'(t) = AX(t)$, but we cannot generally solve the non-autonomous linear differential system, $X'(t) = A(t)X(t), X \in \mathbb{R}^n$. In 1948, T. Wazewski [20] obtained the lower and upper bounds of solutions of the differential equation:

$$|X(t_0)|\, e^{\int_{t_0}^{t} \lambda(s)ds} \leq |X(t)| \leq |X(t_0)|\, e^{\int_{t_0}^{t} \Lambda(s)ds}, X \in \mathbb{R}^n, \tag{1}$$

where $\lambda(t)$ is the minimum value of eigenvalues of $A(t)$, and $\Lambda(t)$ is the maximum value of eigenvalues of $A(t)$. This result is clean, general, optimal, and broadly recognized [13]. In 1987, Wang [15] generalized the linear differential system to nonlinear differential system, $X'(t) = F(t, X(t))G(X(t))$. In 2000, Wang [16] obtained results of Wazewski-type upper and lower bounds of solutions for the linear Volterra integro-differential system

$$X'(t) = A(t)X(t) + \int_{t-h}^{t} B(t,s)X(x)ds + H(t), X \in \mathbb{R}^n.$$

In 2013, Wang [17] obtained results of Wazewski-type upper and lower bounds of solutions for the nonlinear Volterra integro-differential system

$$X'(t) = A(t)F(X(t)) + \int_{t-h}^{t} B(t,s)G(X(x))ds + H(t), X \in \mathbb{R}^n.$$

This research will continue to study Wazewski-type upper and lower bounds of solutions of different kinds of equations, such as nonlinear nonautonomous systems, $X'(t) = F(t, X(t))$, functional differential equations, $X'(t) = F(t, X_t)$, partial differential equations, integral equations, difference equations and function equations. Our goal is to get results as optimal as Wazewskiâ's Inequality (1). Methods of research include Lyapunov's second method, comparison methods, integral and differential inequalities, matrix theory, and mathematical analysis methods.

0.0.2 Gronwall-Type Differential and Integral Inequalities

In 1919, Gronwall [6] gave the following lemma for a system of differential equations with a parameter: Let α, a, b and h be nonnegative constants, and $u : [\alpha, \alpha + h] \to [0, \infty)$ be continuous.

$$\text{If} \quad 0 \le u(t) \le \int_{\alpha}^{t} [bu(s) + a]ds, \alpha \le t \le \alpha + h, \quad \text{then} \quad 0 \le u(t) \le ahe^{bt}, \alpha \le t \le \alpha + h.$$

Bellman generalized it in 1943, which is now commonly known as Gronwall's Inequality, or Gronwall-Bellman's Inequality. This version of Gronwall's inequality is widely applied, and can be found in many references, for example [1,2,4,5,11]: Let α, β and c be nonnegative constants, and $u, f : [\alpha, \beta] \to [0, \infty)$ continuous.

$$\text{If} \quad u(t) \le c + \int_{\alpha}^{t} f(s)u(s)ds, \text{ where } \alpha \le t \le \beta, \quad \text{then} \quad u(t) \le ce^{\int_{\alpha}^{t} f(s)ds}, \alpha \le t \le \beta.$$

In 1958, Bellman generalized his own result by allowing that c to be a nonnegative and nondecreasing function [3,7,11]. Since then, Gronwall's inequality has been found very useful in research of boundedness and stability of differential equations, and error estimates of numerical solutions of differential equations. In 1998, Pachpatte published a book [11] summarizing the development of the Gronwall type inequalities up to 1998. Some recent examples of numerous new generalizations of the Gronwall type can be found in [9, 12, 14, 17]. In 2017, Wang obtained results [18] for the following inequality:

$$W(u(t)) \le g(t) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s) \prod_{j=1}^{m} H_{ij}(u(s)) G_{ij}(\max_{s-h \le \xi \le s}(\xi))ds.$$

0.0.3 Applications of Differential Equations

There are numerous applications in differential equations. Students will apply results of Wazewski-type upper and lower bonds and Grnowall-type differential and integral inequalities on real world applications. For example, they may study stability and periodic solutions of the mosquito population model [8]: Let $w(t)$ and $g(t)$ be the number of wild mosquitoes and sterile mosquitoes at time t.

$$\begin{aligned} \frac{dw}{dt} &= \left[\frac{aw}{1+w+g} - (\mu_1 + \xi_1(w+g)\right] w \\ \frac{dg}{dt} &= \frac{bw}{1+w} - [\mu_2 + \xi_2(w+g)]g \end{aligned}.$$

We will also study a financial stock market model [19]. In the financial stock market, the simple moving average (SMA) of N days is the arithmetic average price of a financial stock for N days, that is, the sum of the price of the stock for N days is divided by$N$. The closing price of a stock is commonly used for calculating the moving average. Let $x(t)$ be the daily closing price of a stock, then the SMA of N days is $\frac{1}{N}\int_{t-N}^{t} x(s)ds$. The following is an example of the model, $x'(t) = \alpha \left(x(t) - \frac{1}{N}\int_{t-N}^{t} x(s)ds\right)$. We understand that the price of a stock can change quickly depending on news and rumors circulated in the financial market. Thus, it may not be possible to model the price of a stock well. Although there has been numerous simulations and time series models of a stock, no continuous model has been studied or published. With little research of this kind, it is not our intention to predict the market. Our goal is to develop mathematical models by functional differential equations that can illustrate technical analysis and market observations [10], and, in a short period, the models can look similar to the curve of S&P 500 or the Dow Jones Industries Average. Students will solve the models, and study the properties of the model such as boundedness and stability of solutions, and existence of periodic solutions.

0.0.4 Computations of Differential Equations

Results and methods on Wazewski type upper and lower bound of differential equations and Gronwall type inequalities can be used to study stability and estimate errors of numerical solutions of differential equations. We expect to obtain better stability results and generate a more accurate error estimate of numerical solutions based on the upper and lower bounds and inequalities we obtained. Study of the stock model requires to find solutions with the stock price of N days. A computer algebra system (CAS), for example, Mathematica, will be used to find and graph solutions of the stock model by the method of steps. We will also use a CAS to solve and graph the mosquito model.

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